Digital Signal Processing (DSP)

Typical analogue and digital signals are shown in figure 66 and figure 67. The lectures up to this point covered analogue systems where the signals are continuous in time. DSP uses discrete signals of the type shown in figure 67. Extra circuitry is required for every operation analogue to digital (ADC) and digital to analogue (DAC) conversion.

Applications

DSP is applied in many areas of the telecommunications, biomedical, and automobile engineering. Typical applications include modems, echo cancellation, digital filtering, modulation, video and image enhancement, CD and DVD, speech and compression for increased storage space, speech synthesis and speech recognition. Many stand-alone DSP integrated circuits operate independently of a PC (they contain their own microprocessor).

Advantages and disadvantages of DSP

Advantages to using DSP techniques are:

- Reproducibility,
- Programmability (flexibility). DSP is flexible since the digital processing can often be easily modified by programming,
- Stability and high reliability: Absence of component drift problems allows for complex processing than is possible with analogue circuitry. DSP provides better signal quality and repeatable performance resulting in lower costs for equivalent performance.
DSP techniques are limited, at present, to signals with relatively low bandwidths (5 MHz video bandwidth). The point at which DSP becomes too expensive will depend on the application and the current state of the processing technology. The cost of high-speed ADC and DAC devices and the extra circuitry required to implement high-speed designs, makes DSP impractical and uneconomical for many applications such as simple filters. Higher power consumption and size of a DSP implementation may make it unsuitable for small-size applications.

**Basic block diagram for a DSP system.**

The block diagram in figure 68 illustrates the main signal processing involved in DSP.

![Basic block diagram for a DSP system.](image)

- An anti-aliasing low-pass filter.
- An A/D converter whose sampling period \( T = \frac{1}{f_s} \). Here \( f_s \) is the sampling frequency with \( f_s \geq 2f_m \). Here \( f_m \) is the highest frequency present in the input signal. Aliasing will occur if the sampling frequency is less than twice the highest frequency contained in the signal.
- This is the digital signal processor
- D/A Converter
- Reconstruction filter

**Sampling Theorem and Aliasing**

All signals sampled in a DSP system require the sampling to take place at a specific rate. The sampling theorem specifies the minimum sample rate required. The original analogue signal may be recovered completely, if a signal, of bandwidth \( f_m \text{ max} \), is sampled at \( f_s \), which is more than \( 2f_m \) (the Nyquist rate). Passing the samples through an ideal low-pass reconstruction filter will eliminate all but the original signal. If the signal to be sampled is not band-limited to a bandwidth of \( f_m \text{ max} \), then frequency components above \( f_s = 2f_m \) are aliased or folded-back and will appear in the sampled signal. These aliased components cause distortion because extra frequency components are present. Aliasing will always be present since practical filters cannot implement the ideal, brick-wall amplitude response, but the idea is to reduce it to acceptable levels. The aliased frequency components are attenuated using an anti-aliasing analogue filter before sampling.
The amount of aliasing, which may be tolerated, depends on the filter design. A filter, whose response at the maximum desired frequency, is one to two percent of the passband signal level, will achieve good results. The filter should attenuate aliased frequency components to a comparable level equal to the quantisation noise level.

Quantization Noise and Dynamic Range

The dynamic range of the ADC is the ratio of the maximum to minimum signal levels. Expressed in dB is

\[ D = 20 \log \frac{V_{\text{max}}}{V_{\text{min}}}. \]

Quantisation noise is produced by the ADC process and corrupts the signal. This is because the analogue signal must be rounded (truncated) to the nearest quantised value. For a sinusoidal input signal at the maximum ADC input level, the signal-to-quantisation noise ratio (SQNR) in dB, for a \( n \)-bit ADC, is:

\[ \text{SQNR} = 6n + 1.8 \ (6 \text{ dB rule}). \]

The original signal spectrum appears at multiples of the sampling frequency but may be extracted from the sampled spectrum by passing the sampled signal through a low-pass filter. The cut-off frequency of the filter should equal the maximum frequency of the original signal. A typical DAC outputs a step waveform instead of the theoretical impulse train. This zero order hold circuit has the same effect as passing the signal through a filter whose impulse response is a pulse whose width is that of the sampling period. The transfer function of the filter amplitude response has a \( \sin x/x \) shape, which should be taken into account. Adding a filter with an inverse \( x/\sin(x) \) amplitude response in the DSP processor or the reconstruction filter, should compensate for this effect. The reconstruction filter is similar to the anti-aliasing filter and designed so that the aliased frequency components are reduced to acceptable levels.

Digital Signals

Continuous time signals (analogue signal) exist for all values of time. However, discrete time signals are defined at discrete values of time only. For ease of computation these time instance are equally spaced. Consider an analogue signal \( v \) connected as shown in figure 69. The signal path is interrupted by a switch, which is closed and opened at the sampling rate.

![Figure 69: A sampler.](image)
We can for convenience refer to $x(nT)$ as $x(n)$ since $T$ is constant. It is important to recognise that $x(n)$ is only defined for integer values of $n$. It is not right to think of $x(n)$ as being zero for $n$ not being an integer, but simply undefined for non-integer values of $n$. Some important DSP signals are now considered.

**Unit Step**

![Unit Step signal](image1)

**Figure 70: Unit Step function.**

The unit step signal is normally given the symbol $u(n)$

$$u(n) = 1 \text{ for } n \geq 0$$

$$u(n) = 0 \text{ for } n < 0$$

Note the signal is not zero before time $= 0$ but is just not defined at that time period.

**Unit Impulse**

![Unit Impulse signal](image2)

**Figure 71: Unit Impulse function.**

$$\delta(n) = 1 \text{ for } n = 0$$

$$\delta(n) = 0 \text{ elsewhere}$$
It is important to understand that the impulse only exists at $n = 0$. The unit impulse $\delta(n)$, may be expressed in terms of the step function as:

$$\delta(n) = u(n) - u(n - 1)$$  \hspace{1cm} (3)

**Decaying exponential**

$$G(n) = a^n \text{ for } n \geq 0$$  \hspace{1cm} (4)

$$G(n) = 0 \text{ for } n < 0$$

Where $0 < a < 1$ i.e. $a$ is a fractional number.

---

**Discrete Sinusoidal Signal**

The analogue sine signal is

$$x(t) = A \sin \omega_0 t = A \sin 2\pi f_0 t$$  \hspace{1cm} (5)
The discrete form

\[ x(n) = A \sin(\omega_a nT) \]  \hspace{1cm} (6)

Now \( T = \frac{1}{f_s} \) is the sampling period and equal to the inverse of the sampling frequency \( n \). So

\[ x(n) = A \sin(2\pi f_a \frac{n}{f_s}) = A.\sin(2\pi \frac{f_a}{f_s} n) \]  \hspace{1cm} (7)

The factor \( 2\pi \frac{f_a}{f_s} = \theta \) is called the digital frequency. Therefore

\[ x(n) = A.\sin(n\theta) - \pi \leq \theta \leq \pi \]  \hspace{1cm} (8)

A digital signal is said to be periodic with period \( N \) if \( N \) is the smallest integer for which \( x(n + N) = x(n) \)

\[ A \sin(n + N)\theta = A \sin n\theta \]

Which can only be satisfied for all \( n \) if:

\[ N\theta = 2\pi k \Rightarrow N = \frac{2\pi k}{\theta} = \frac{f_s}{f_a} \times k \]

Where \( k \) is any constant

**Example 1**

If the analogue signal is \( f_a = 1 \) kHz, and the sampling frequency is \( f_s = 8 \) kHz, then we can write

\[ N = \frac{8 \text{ kHz}}{1 \text{ kHz}} = 8 \text{ samples} \]

Consider two sampled signals shown in figure 74 and figure 75.
Transformation of Digital Signals

A digital signal $x(n)$ may be shifted in time (advanced or delayed) by replacing the variable $n$ with $n - k$ where $k$ is an integer greater than zero, so that $x(n - k)$ is the signal $x(n)$ delayed by $k$ samples and $x(n + k)$ is $x(n)$ advanced by $k$ samples. See figure 76.

Figure 76: Advanced and delayed signals

To represent an arbitrary sequence we can express it as the sum of scaled delayed unit impulses

$$y(n) = a_1 x(n+1) + a_2 x(0) + a_3 x(n-1) - a_4 x(n-2)$$  \hspace{1cm} (9)

Where $a_1, a_2, a_3, a_4$ are the weightings or magnitude values and $x$ is a unit impulse signal.

$$y(n) = 0.2 \delta(n+1) + 0.1 \delta(0) + 0.5 \delta(n-1) - 0.75 \delta(n-2)$$  \hspace{1cm} (10)

Figure 77: Delayed signals.

More generally, arbitrary sequences may be expressed using the following formula:

$$y(n) = \sum_{k=-\infty}^{\infty} a(k) x(n-k)$$  \hspace{1cm} (11)
This formula for real signals is written:

$$y(n) = \sum_{k=0}^{\infty} a(k)x(n-k)$$  \hspace{1cm} (12)

We can also express equation (12) if we have a finite number of samples $N$ as

$$y(n) = \sum_{k=0}^{N} a(k)x(n-k)$$  \hspace{1cm} (13)

Equation (13) is a special form of the very important topic of convolution examined in the next section.

**Linear time-invariant (LTI) systems**

A discrete time system is an algorithm that maps a discrete signal or input signal $x(n)$ onto a discrete output signal $y(n)$. For a system to be linear:

$$a_1 x_1(n) + a_2 x_2(n) \rightarrow b_1 y_1(n) + b_2 y_2(n)$$  \hspace{1cm} (14)

(no extra terms generated by the system and is therefore linear) where $a_1$, $a_2$, $b_1$, $b_2 = \text{arbitrary constants}$. A time-invariant system is defined as

$$x(n-n_o) \rightarrow y(n-n_o)$$

The system (or transfer) function is:

$$H = \frac{y(n)}{x(n)}$$

(Both signals delayed by same amount). These two properties must be present if the system is to be LTI.

**Stability of a DSP System: Bounded Input, Bounded Output. (BIBO)**

a) A signal $x(n)$ is bounded if there exists a finite $M$ such that the magnitude of $x(n)$ is less than $M$ for all values of $n$

$$|X(n)| < M \ (\text{for all } n)$$

b) A system is BIBO stable if every bounded input $x(n)$ produces a bounded output sequence

c) A discrete time sequence is called causal if it has zero values for $n<0$

d) A LTI system with impulse response $h(n)$ is causal if and only $h(n)$ is zero for $n<0$

e) If the impulse response of an LTI system is of finite duration, the input system is said to be a finite impulse response system (FIR) BIBO stable

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f) If the impulse response of an LTI system is of infinite duration, the system is an infinite-impulse response system (IIR).

**Filters**

The output signal from digital filters, can be characterised according to the equation:

\[
y(n) = \sum_{k=0}^{m} a(k)x(n-k) - \sum_{k=1}^{l} b(k)y(n-k)
\]  

(15)

We can use this definition to classify FIR and IIR type filters. A simple block diagram, using a first order representation for each filter, is shown in figure 80 and figure 81.

**Block diagram form for representing difference equations**

The following blocks will represent a system implementation of a particular system:

![Figure 78: Summer](image)

![Figure 79: Multiplier](image)

![Figure 80: Delay](image)

**Example 2**

Consider the following difference equation, which represents a FIR filter:

\[
y(n) = x(n) + 0.5x(n-1)
\]  

(16)

![Figure 81: FIR filter with a = 0.5.](image)

**System Implementation**

\[
y(n) = x(n) - x(n-2) + 1.27y(n-1) - 0.81y(n-2)
\]  

(17)

Draw a system implementation. Find the difference equation from the system implementation below. This represents a IIR filter because of the feedback from output to input.
Exercise 3

The output signal from digital filters is characterised according to the equation:

\[ y(n) = \sum_{k=0}^{m} a(k)x(n-k) - \sum_{k=1}^{L} b(k)y(n-k) \]

Use this expression to classify digital filters into FIR and IIR type filters. Give a simple block diagram using a first order representation for each filter. Draw the block diagram of the following difference equations:

\[ y(n) = x(n) + 0.047x(n-1) - 0.453y(n-1) - 0.207y(n-3) \]

\[ y(n) = a_0x(n) + a_1x(n-1) + a_2x(n-2) + b_1y(n-1) + b_2y(n-2) \]

Impulse Response

The output response \( y(n) \) when we apply an input signal \( x(n) = \delta(0) \), an impulse, is known as the impulse response and given the name \( h(n) \). This is a very important characteristic of a discrete system. The output tells us the system behaviour as the system is being hit by all input frequencies, that is \( h(n) \) completely characterises the system.

The Z-plane and the Z-Transform:

The Laplace transform was covered in the previous chapter and now we introduce the Z transform and perform similar analysis and concepts in the Z-plane.

The Z transform:

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \]  

(18)

Here \( Z \) is defined:

\[ Z = e^{iT} = e^{j\omega T} \]  

(19)

For a one-sided Transform
A sampled data signal is composed of a sequence of numerical values, which represent the amplitude of the signal at the instance of sampling. Since the time between sampling is constant we can use the Z transform to model time delays for example a time delay of 1 unit, we simply substitute $Z^{-1}$ into the appropriate equations. That is a unit delay in the time domain is equivalent to multiplication by $Z^{-1}$ in the Z domain.

**Z transform of some useful sequences:**

1) Unit Impulse:

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} \delta(n)Z^{-n}$$

$$x(n) = \delta(n) = 1$$

Consider the values of $n = 0, 1, 2, 4...$ etc.

For $n = 0$

$$X(z) = 1 Z^{-0} = 1$$

For $n = 1$

$$X(z) = 0 Z^{-1} = 0$$

All other values are zero since $X(z)$ has a value of 1 only at $n = 0$

$$X(z) = 1$$

(21)

2) A unit step:

$$x(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

We can represent the unit step as a sum of delayed Impulses so that we can write the Z - transform of a unit step as:

$$X(z) = \sum_{n=0}^{\infty} x(n)Z^{-n} = 1 + 1.Z^{-1} + 1.Z^{-2} + ...$$

$$X(z) = 1 + Z^{-1} + Z^{-2} + ....$$

See figure 70.
An arithmetic series may be written in closed form as:

\[
\frac{1}{1-Z^{-1}} x \frac{Z}{Z} = \frac{Z}{Z-1}
\]  

(22)

Consider the progression

\[ S = a + ar + ar^2 + ar^3 + \ldots \]

Multiply by \( r \)

\[ rS = ar + ar^2 + ar^3 + ar^4 + \ldots \]

And subtracting yields

\[ S - rS = a \]

\[ S = \frac{a}{1-r} \]

Where \( a = 1 \) and \( r = Z^{-1} \)

3) Unit delay: If we consider a delayed unit impulse, we can represent this in a time diagram. Delayed unit impulse at \( n = 4 \) or in general we can define it as:

\[ x(k) = 1 \text{ for } n = k \text{ but } 0 \text{ elsewhere} \]

Thus we can apply the \( Z \)-transform to this delayed impulse.

\[ X(z) = \sum_{n=0}^{\infty} x(k)Z^{-n} = 1.Z^{-4} + 0 + 0. \]

Where \( k \) is this particular delay

\[ x(n) = \delta(n - 3) \]

\[ X(z) = 1Z^{-3} \]

If \( x(n) = \delta(n - 1) \), then

\[ X(z) = 1Z^{-1} \]

4) Exponential Series

\[ x(n) = \begin{cases} a^n & n \geq 0 \\ a^n 0.n < 0 \end{cases} \]

\[ \ldots a < 1 \]
Pole-Zero

The definition of pole and zeros in the Laplace section can be applied in the same manner to the Z-domain. A pole is defined as that value of $Z$ in the denominator, which causes the system function to have an infinite value. The zero is that value of $Z$ in the numerator, which cause the system function to go to zero. Consider the FIR first-order system whose difference function is:

$$y(n) = x(n) + 0.5x(n-1)$$  \hspace{1cm} (23)

Expressed in terms of the $Z$-transform:

$$Y(z) = X(z) + 0.5 X(z) Z^{-1} = X(z)[1 + 0.5 Z^{-1}]$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + 0.5 Z^{-1} = \frac{Z + 0.5}{Z}$$ \hspace{1cm} (24)

The factor $(1 + 0.5 Z^{-1})$ is manipulated into a suitable form by multiplying above and below by $Z$. We can then see that there is a zero at $Z = -0.5$ (A value which causes the numerator to have a zero value—definition) and a pole at $Z = 0$. The pole-zero map is shown in figure 82. Here the pole is represented on the unit circle by the symbol ‘x’ and a zero, by a small circle. The $Z$-transform has a unity magnitude and hence the $Z$-plane is bounded by unity.

![Figure 83: Pole-zero plot.](image)

Stability

If the poles of a transfer function are contained within the unit circle, the system is stable. If the poles are located on the unit circle, then the system has marginal stability.

Example 4

(a) Determine $H(z)$ for the following
(i) \( y(n) = x(n) - a_o \cdot x(n - 1) \)

(ii) \( y(n) = (1 - a_o) \cdot x(n) + a_o \cdot y(n - 1) \)

(iii) \( y(n) = x(n) + a_o \cdot x(n) + a_2 \cdot x(n - 2) - b_1 \cdot y(n - 1) - b_2 \cdot y(n - 2) \)

(b) Obtain the difference equations from the system functions:

\[
H(z) = \frac{1}{1 - 1.2Z^{-1} + 0.91Z^{-2}}
\]

\[
H(z) = \frac{1-Z^{-2}}{1-1.2Z^{-1} + 0.81Z^{-2}}
\]

**Solution**

(a)

(i) \( Y(z) = X(z) - a_o \cdot X(z) \cdot Z^{-1} \)

\( Y(z) = X(z)[1 - a_o \cdot Z^{-1}] \)

\( H(z) = 1 - a_o \cdot Z^{-1} \)

(ii) \( Y(z) = (1 - a_o) \cdot X(z) + a_o \cdot Y(z) \cdot Z^{-1} \)

\( Y(z)[1 - a_o \cdot Z^{-1}] = (1 - a_o) \cdot X(z) \)

\( H(z) = \frac{1 - a_o}{1 - a_o \cdot Z^{-1}} \)

(iii) \( Y(z) = X(z) + a_1 \cdot X(z) + a_2 \cdot X(z) \cdot Z^{-2} - b_1 \cdot Y(z) \cdot Z^{-1} - b_2 \cdot Y(z) \cdot Z^{-2} \)

\( Y(z)[1 + b_1 \cdot Z^{-1} + b_2 \cdot Z^{-2}] = X(z) \cdot [1 + a_o + a_2 \cdot Z^{-2}] \)

\( H(z) = \frac{1 + a_1 + a_2 \cdot Z^{-2}}{1 + b_1 \cdot Z^{-1} + b_2 \cdot Z^{-2}} \)

(b)

(i) \( \frac{Y(z)}{X(z)} = \frac{1}{1 - 1.2Z^{-1} + 0.91Z^{-2}} \)

\( Y(z) - 1.2Y(z)Z^{-1} + 0.91Y(z)Z^{-2} = X(z) \)

\( y(n) = x(n) + 1.2y(n - 1) - 0.91y(n - 2) \)
(ii)

\[
\begin{align*}
\frac{Y(z)}{X(z)} &= \frac{1 - Z^{-2}}{1 - 1.2Z^{-1} + 0.81Z^{-2}} \\
y(n) &= x(n) - x(n-2) + 1.2y(n-1) - 0.81y(n-2)
\end{align*}
\]

c) factorise (b) (i) and (b) (ii)